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EXAMPLES OF OSSERMAN METRICS OF (3, 3)-SIGNATURE

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Abstract

In this paper, the Osserman condition is studied for a family of affine torsion free connection. As applications, we provide examples of Osserman pseudo-Riemannian metrics of signature (3, 3) on the cotangent bundle of a manifold.

1. Introduction

Let (M, g) be a pseudo-Riemannian manifold of signature (p, q) and dimension $m = p + q$. Let ∇ be the Levi-Civita connection defined by the metric g, and let $\mathcal{R}(X, Y) := \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X, Y]}$ be the curvature

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operator. The Jacobi operator $J_{\mathcal{R}}(X) : Y \to \mathcal{R}(Y, X)X$ is a self-adjoint operator and it plays an important role in the curvature theory. Let $Spect\{J_{\mathcal{R}}(X)\}$ be the set of eigenvalues of Jacobi operator $J_{\mathcal{R}}(X)$ and $S^{\pm}(M, g)$ be the pseudo-sphere bundles of unit spacelike (+) and timelike (–) tangent vectors. One says that (M, g) is *spacelike Osserman* (*respectively*, *timelike Osserman*) at $P \in M$, if for every $X, Y \in S_P^+(M, g)$ (respectively, for every $X, Y \in S_P^-(M, g)$), $Spect\{J_{\mathcal{R}}(X)\} = Spect\{J_{\mathcal{R}}(Y)\}$. One says that (M, g) is pointwise spacelike Osserman (respectively, pointwise timelike Osserman), if it is spacelike Osserman (respectively, timelike Osserman) at every point of M and (M, g) is globally spacelike Osserman (respectively, globally timelike Osserman), if the eigenvalues structures does not in fact depend on the point under consideration. The notions of spacelike Osserman and timelike Osserman are equivalent and if (M, g) is either of them, then (M, g) is said to be *Osserman*. The investigation of Osserman manifolds has been an extremely attractive and fruitful one in recent years; we refer to [3, 5, 6] for further details.

This paper aims to generalize these notions to the affine geometry. The main result is the local description of a family of the affine Osserman on 3-dimensional manifolds. As an application, we provide some explicit examples of Osserman metric of signature (3, 3) by the construction so-called *Riemann extension*. This construction, which relates affine and pseudo-Riemannian geometries, associates a neutral signature on T^*M to any torsion for connection ∇ on the base manifold M. Riemann extensions have been used both to understand questions in affine geometry and to solve curvature problems. For instance, it is known that (M, ∇) has zero curvature, if and only if $(T^*M, \overline{g})$ has zero curvature. Also, (M, ∇) is locally symmetric, if and only if $(T^*M, \overline{g})$ is locally symmetric [7]. Further, (M, ∇) is Osserman, if and only if the Riemann extension is Osserman [4].

Our paper is organized as follows. Section 1 introduces these topics. In Section 2, we recall some basic definitions about affine Osserman and the Riemann extension. In Section 3, we prove the following result:

Theorem 1.1. *Let M be a* 3*-dimensional manifold with torsion free connection given by*:

$$\begin{cases} \nabla_{\partial_1}\partial_1 = f_1(u_1, u_2, u_3)\partial_1; \\ \nabla_{\partial_2}\partial_2 = f_2(u_1, u_2, u_3)\partial_2; \\ \nabla_{\partial_1}\partial_3 = f_3(u_1, u_2, u_3)\partial_2. \end{cases} \tag{1}$$

Then (M, ∇) *is affine Osserman, if and only if the functions* f_1, f_2, f_3 *are as follows satisfy the following PDE's*:

$$\begin{cases} f_1(u_1, u_2, u_3) = A(u_1, u_2); \\ f_2(u_1, u_2, u_3) = B(u_1, u_2); \\ f_3(u_1, u_2, u_3) = k(u_3)e^C(u_1, u_2); \end{cases}$$

where k is a function and A, B, and C subject the following PDE:

$$\partial_2 A + \partial_1 B = 0, \textit{ where } \partial_2 C = -B \textit{ and } \partial_1 C = A.$$

2. Preliminaries

2.1. Affine Osserman manifolds

Let (M, ∇) be an affine manifold, where ∇ is a torsion free connection on *TM*. Let $\mathcal{R}^\nabla$ be the curvature operator and $J_{\mathcal{R}^\nabla}(X):$ $Y \to \mathcal{R}^\nabla(Y, X)X$ be the *affine Jacobi operator*; we will write $\mathcal{R}^\nabla$ and $J_{\mathcal{R}^\nabla}$ when it is necessary to distinguish the role of the connection. One says that (M, ∇) is *affine Osserman*, if $J_{\mathcal{R}^\nabla}(X)$ is nilpotent for all tangent vectors *X*, i.e., $Spect\{J_{\mathcal{R}^\nabla}(X) = 0\}$.

The authors in [4] pay special attention to dimension $m = 2$. In that case, they proved that ∇ is affine Osserman, if and only if the Ricci tensor of ∇ is skew-symmetric on *M*. They also gave a description of Osserman connections on a 2-dimensional manifold.

Theorem 2.1 ([4]). *Let M be a 2-dimensional manifold with torsion free affine Osserman connection ∇. Then, at each $p \in M$, either the Ricci tensor of ∇ vanishes or there is a system of coordinates (u_1, u_2) in which, the nonzero components of the connection are*

$$\text{(i)} \qquad \Gamma_{11}^{1} = -\partial_1 \theta, \quad \Gamma_{22}^{2} = \partial_2 \theta,$$

where θ is a smooth function such that $\partial_1 \partial_2 \theta \neq 0$; or

$$\text{(ii)} \qquad \Gamma_{22}^{1} = \varphi, \quad \Gamma_{11}^{1} = -\partial_1 \log \varphi, \quad \Gamma_{22}^{2} = \partial_2 \log \varphi,$$

where φ is a smooth function such that $\partial_1 \partial_2 \log \varphi \neq 0$; or

$$\text{(iii)} \qquad \Gamma_{22}^{1} = -\psi / (1 + u_1 u_2), \quad \Gamma_{11}^{2} = 1 / [\psi(1 + u_1 u_2)],$$

$$\Gamma_{11}^{1} = -\partial_1 \log \psi + u_2 / (1 + u_1 u_2),$$

$$\Gamma_{22}^{2} = \partial_2 \log \psi + u_1 / (1 + u_1 u_2),$$

where ψ is a smooth function such that $\partial_1 \partial_2 \log \psi \neq 0$.

The authors of [4] used the connections of the simplest type (i) to build up examples of pseudo-Riemannian Osserman manifolds of signature (2, 2).

For dimension $m = 3$, to make a description seems to be a hard problem. The aim of the present paper is to describe the connection (1), which are *affine Osserman*.

One has the following observation:

Theorem 2.2. *Let (M_1, ∇_1) be affine Osserman at $P_1 \in M_1$ and (M_2, ∇_2) be affine Osserman at $P_2 \in M_2$. Then, the product structure $M := (M_1 \times M_2, \nabla_1 \oplus \nabla_2)$ is affine Osserman at $P = (P_1, P_2)$.*

Proof. If $X = (X_1, X_2) \in T_{(P_1, P_2)}(M_1 \times M_2)$, then

$$J_{\mathcal{R}^{\nabla}}(X) = J_{\mathcal{R}^{\nabla_1}}(X_1) \oplus J_{\mathcal{R}^{\nabla_2}}(X_2).$$

So

$$Spect\,\{J_{\mathcal{R}^{\nabla}}(X)\} = Spec\,\{J_{\mathcal{R}^{\nabla_1}}(X_1)\} \cup Spec\,\{J_{\mathcal{R}^{\nabla_2}}(X_2)\}$$

$$= \{0\} \cup \{0\} = \{0\}. \qquad \square$$

2.2. Riemann extension construction

Let (M, ∇) be a 3-dimensional affine manifold. Let (u_1, u_2, u_3) be the local coordinates on M. We expand $\nabla_{\partial_i}\partial_j = \sum_k \Gamma_{ij}^k \partial_k$ for $i, j, k = 1, 2, 3$ to define the Christoffel symbols of ∇. Let $\omega = u_4 du_1 + u_5 du_2 + u_6 du_3 \in T^*M : (u_4, u_5, u_6)$ are the dual fiber coordinates. The *Riemann extension* of the connection (1) is the pseudo-Riemannian metric $\bar{g}$ on the cotangent bundle T^*M of neutral signature (3, 3) defined by setting:

$$\bar{g}(\partial_1, \partial_4) = \bar{g}(\partial_2, \partial_5) = \bar{g}(\partial_3, \partial_6) = 1,$$

$$\bar{g}(\partial_1, \partial_1) = -2u_4\Gamma_{11}^1 - 2u_5\Gamma_{11}^2 - 2u_6\Gamma_{11}^3,$$

$$\bar{g}(\partial_1, \partial_2) = -2u_4\Gamma_{12}^1 - 2u_5\Gamma_{12}^2 - 2u_6\Gamma_{12}^3,$$

$$\bar{g}(\partial_1, \partial_3) = -2u_4\Gamma_{13}^1 - 2u_5\Gamma_{13}^2 - 2u_6\Gamma_{13}^3,$$

$$\bar{g}(\partial_2, \partial_2) = -2u_4\Gamma_{22}^1 - 2u_5\Gamma_{22}^2 - 2u_6\Gamma_{22}^3,$$

$$\bar{g}(\partial_2, \partial_3) = -2u_4\Gamma_{23}^1 - 2u_5\Gamma_{23}^2 - 2u_6\Gamma_{23}^3,$$

$$\bar{g}(\partial_3, \partial_3) = -2u_4\Gamma_{33}^1 - 2u_5\Gamma_{33}^2 - 2u_6\Gamma_{33}^3.$$

We refer to [7] for more details. We have the following result:

Theorem 2.3 ([4]). *Let $(T^*M, \bar{g})$ be the cotangent bundle of an affine manifold (M, ∇) equipped with the Riemannian extension of the torsion free connection ∇. Then $(T^*M, \bar{g})$ is a pseudo-Riemannian globally Osserman manifold, if and only if (M, ∇) is an affine Osserman manifold.*

The Riemann extension $\bar{g}$ have been used in [4] to construct nonsymmetric Osserman metrics of signature (2, 2).

3. Proof of Theorem

Lemma 3.1. *The components of the curvature operator of the connection* (1) *are given by*:

$$\mathcal{R}^{\nabla}(\partial_1, \partial_2)\partial_1 = -\partial_2 f_1 \partial_1,$$

$$\mathcal{R}^{\nabla}(\partial_1, \partial_2)\partial_2 = \partial_1 f_2 \partial_2,$$

$$\mathcal{R}^{\nabla}(\partial_1, \partial_2)\partial_3 = -(\partial_2 f_3 + f_2 f_3)\partial_2,$$

$$\mathcal{R}^{\nabla}(\partial_1, \partial_3)\partial_1 = -\partial_3 f_1 \partial_1 + (\partial_1 f_3 - f_1 f_3)\partial_2,$$

$$\mathcal{R}^{\nabla}(\partial_1, \partial_3)\partial_3 = -\partial_3 f_3 \partial_2,$$

$$\mathcal{R}^{\nabla}(\partial_2, \partial_3)\partial_1 = (\partial_2 f_3 + f_2 f_3)\partial_2,$$

$$\mathcal{R}^{\nabla}(\partial_2, \partial_3)\partial_2 = -\partial_3 f_2 \partial_2.$$

Lemma 3.2. *If* $X = \sum_1^2 \alpha_i \partial_i$ *is a vector on M, then the affine Jacobi operator is given by*:

$$J_{\mathcal{R}^{\nabla}}(X)\partial_1 = a_1 \partial_1 + b_1 \partial_2,$$

$$J_{\mathcal{R}^{\nabla}}(X)\partial_2 = a_2 \partial_1 + b_2 \partial_2,$$

$$J_{\mathcal{R}^{\nabla}}(X)\partial_3 = a_3 \partial_1 + b_3 \partial_2;$$

where

$$a_1 = -\alpha_1 \alpha_2 \partial_2 f_1 - \alpha_1 \alpha_3 \partial_3 f_1,$$

$$a_2 = \alpha_1^2 \partial_2 f_1,$$

$$a_3 = \alpha_1^2 \partial_3 f_1,$$

$$b_1 = \alpha_2^2\partial_1 f_2 - \alpha_2\alpha_3(\partial_2 f_3 + f_2 f_3) + \alpha_1\alpha_3(\partial_1 f_3 - f_1 f_3) - \alpha_3^2\partial_3 f_3,$$

$$b_2 = -\alpha_1\alpha_2\partial_1 f_2 + 2\alpha_1\alpha_3(\partial_2 f_3 + f_2 f_3) - \alpha_2\alpha_3\partial_3 f_2,$$

$$b_3 = -\alpha_1^2(\partial_1 f_3 - f_1 f_3) + \alpha_1\alpha_3\partial_3 f_3 - \alpha_1\alpha_2(\partial_2 f_3 + f_2 f_3) + \alpha_2^2\partial_3 f_2.$$

The matrix associated to $J_{\mathcal{R}^\nabla}(X)$ with respect to the basis $\{\partial_1, \partial_2, \partial_3\}$ is given by:

$$(J_{\mathcal{R}^\nabla}(X)) = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ 0 & 0 & 0 \end{pmatrix}.$$

It follows from the matrix associated to $J_{\mathcal{R}^\nabla}(X)$, that its characteristic polynomial can be written as follows:

$$P_\lambda[J_{\mathcal{R}^\nabla}(X)] = -\lambda^3 + \lambda^2(a_1 + b_2) + \lambda(a_2 b_1 - a_1 b_2). \tag{2}$$

Lemma 3.3. *Let* (M, ∇) *be a* 3*-dimensional manifolds with a torsion free connection* ∇ *given system* (1). *The* (M, ∇) *is affine Osserman manifolds*, *if and only if the functions* f_1, f_2, f_3 *satisfying the following PDE's*:

$$\begin{cases} \partial_3 f_2 = 0, \quad \partial_2 f_1 \partial_3 f_1 & = 0, \\ \partial_2 f_1 \partial_3 f_2 = 0, \quad \partial_3 f_1 \partial_3 f_2 & = 0, \\ \partial_1 f_2 + \partial_2 f_1 & = 0, \\ \partial_2 f_1(\partial_1 f_3 - f_1 f_3) & = 0, \\ 2(\partial_2 f_3 + f_2 f_3) - \partial_3 f_1 & = 0, \\ (\partial_2 f_1 + 2\partial_3 f_1)(\partial_2 f_3 + f_2 f_3) - \partial_1 f_2 \partial_3 f_1 & = 0. \end{cases} \tag{3}$$

Proof. Considering the characterical polynomial (2) and the Osserman condition in [4] for affine manifolds, then we have

$$\begin{cases} a_1 + b_1 & = 0, \\ a_2 b_1 - a_1 b_2 & = 0. \end{cases}$$

The system (3) of PDE's is obtained by computations of this previous system. $\square$

Proof of Theorem 1.1. From the system (3), we have $\partial_3 f_2 = 0$. Then

$$f_2 = f_2(u_1, u_2, u_3) = B(u_1, u_2). \tag{4}$$

In the same way, we also have $\partial_2 f_1 \partial_3 f_1 = 0$, then

$$\partial_2 f_1 = 0 \quad \text{or} \quad \partial_3 f_1 = 0.$$

Suppose $\partial_2 f_1 = 0$ and $\partial_3 f_1 \neq 0$. Then

$$f_1 = f_1(u_1, u_2, u_3) = A(u_1, u_3). \tag{5}$$

With the last two equations in (3), we have

$$\begin{cases} 2(\partial_2 f_3 + f_2 f_3) & = \partial_3 f_1, \\ 2\partial_3 f_1(\partial_2 f_3 + f_2 f_3) & = 0. \end{cases}$$

It is absurde, then $\partial_3 f_1 = 0$.

Hence

$$f_1 = f_1(u_1, u_2, u_3) = A(u_1, u_2). \tag{6}$$

From

$$\begin{cases} 2(\partial_2 f_3 + f_2 f_3) & = 0, \\ \partial_2 f_1(\partial_2 f_3 + f_2 f_3) & = 0, \\ \partial_2 f_1(\partial_1 f_3 - f_1 f_3) & = 0; \end{cases}$$

we conclude that

$$f_3 = f_3(u_1, u_2, u_3) = k(u_3) e^{C(u_1, u_2)}, \tag{7}$$

where $\partial_2 C(u_1, u_2) = -B(u_1 u_2)$ and $\partial_2 C(u_1, u_2) = A(u_1 u_2)$. $\square$

Example 3.1. The following connection on $\mathbb{R}^3$, which is defined by

$$\nabla_{\partial_1}\partial_1 = u_2 \partial_1; \quad \nabla_{\partial_2}\partial_2 = -u_1 \partial_2; \quad \nabla_{\partial_1}\partial_3 = e^{(u_1 u_2)}\partial_2, \tag{8}$$

is affine Osserman.

Example 3.2. The following connection on $\mathbb{R}^3$, which is defined by

$$\nabla_{\partial_1}\partial_1 = u_1 u_2 \partial_1; \quad \nabla_{\partial_2}\partial_2 = -\frac{1}{2} u_1^2 \partial_2; \quad \nabla_{\partial_1}\partial_3 = e^{\left(\frac{1}{2} u_1^2 u_2\right)} \partial_2, \tag{9}$$

is affine Osserman.

Corollary 3.1. *Let* $\mathbb{R}^3$ *and* ∇ *be the torsion free connection given by* (8) *or* (9). *Let* $\overline{g}$ *be the Riemann extension on* T^*M. *Then* $\overline{g}$ *is an Osserman metric of signature* (3, 3).

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